Example of operads: the genus zero modular operad

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Definition with vector spaces

- A (symmetric) operad $P$ consists of a collection of $k$-vector spaces $\{P(n)\}_{n \geq 1}$ (such that the symmetric group $\mathbb{S}_r$ acts on $P(r)$) endowed with:
  
  ▶ composition maps $\circ_i : P(k) \times P(l) \to P(k+l-1)$,
  
  ▶ a unit morphism $\eta : 1 \to P(1)$,

satisfying some axioms (equivariance, unit, associativity).

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  - composition maps $\circ_i : P(k) \times P(l) \to P(k + l - 1)$,
  - a unit morphism $\eta : 1 \to P(1)$
  - + satisfying some axioms (equivariance, unit, associativity).

Example of operads: the genus zero modular operad
Operads can be applied everywhere ...  
... as long as you have a 

**symmetric monoidal category.**
Symmetric monoidal category

Ingredients:

1. Category $\mathcal{C}$,
Symmetric monoidal category

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1. Category $\mathcal{C}$,
2. a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
3. a unit object $1 \in \mathcal{C}$,
4. natural isomorphisms $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$
5. coherence axioms,
6. and symmetry isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ such that $c_{X,Y}c_{Y,X} = id$.

Example of operads: the genus zero modular operad
Symmetric monoidal category

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1. Category $C$
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Example of operads: the genus zero modular operad
Symmetric monoidal category

Ingredients:

1. Category \( \mathcal{C} \),
2. a tensor product \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \),
3. a unit object \( \mathbf{1} \in \mathcal{C} \),
4. natural isomorphisms \( (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \)
5. coherence axioms,
6. and symmetry isomorphisms \( c_{X,Y} : X \otimes Y \to Y \otimes X \) such that \( c_{X,Y} c_{Y,X} = \text{id} \).
We can think of an $n$-ary operation as a little black box with $n$ wires coming in and one wire coming out:
Shrink the black box to a point, you obtain this graph ᵃ:
Trees for operads

**Tree** $T$:

- non-empty, connected graph.
- No loops.
- Can be oriented.

**Property:** At each vertex there exists at least one incoming edge; exactly one outgoing edge.

**External edge:** bounded by a vertex (one end only).

**Internal edges:** those bounded by vertices at both ends (all edges that are not external).
Any tree has:
- a **unique outgoing** external edge, called the **output** (or the **root**) of the tree,
- **several ingoing** edges, called **inputs** or leaves of the tree.

Similarly, the edges going in and out of a vertex $v$ of a tree will be referred to as inputs and outputs at $v$. 
Operadic zoo

Algebras:

- Operad

Graphs:

- Rooted trees

Example of operads: the genus zero modular operad
Operadic zoo

Algebras:

- Operad
- Cyclic operads

Graphs:

- Rooted trees
- Trees

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Operadic zoo

Algebras:

- Operad
- cyclic operads
- k-modular

Graphs:

- Rooted trees
- trees
- connected + orientation + on set of edges + genus marking

Example of operads: the genus zero modular operad
Operadic zoo

Algebras:

- Operad
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- k-modular
- dioperads

Graphs:

- Rooted trees
- trees
- connected + orientation + on set of edges + genus marking
- connected directed graphs w/o directed loops or parallel edges

Example of operads: the genus zero modular operad
Operadic zoo

Algebras:

- Operad
- cyclic operads
- k-modular
- dioperads
- properads

Graphs

- Rooted trees
- trees
- connected + orientation + on set of edges + genus marking
- connected directed graphs w/o directed loops or parallel edges
- connected directed graphs w/o directed loops
Borisov–Manin’s generalized operad definition

Definition [Borisov–Manin]

Operads of various types are certain functors from a category of labeled graphs $\Gamma$ to a symmetric monoidal category $(G, \otimes)$ which will be called ground category.

Example

The simplest example is that of finite-dimensional vector spaces over a field, or that of finite complexes of such spaces. 

*N.B*: The word 'operad' is in the wide sense (i.e. May and Markl operads, cyclic operads, modular operads, PROPS, properads, dioperads etc.).
Operadic zoo:
What kind of operadic creatures can we find?

**Modular operad. No distinction** between *inputs* and *outputs*.

**EXAMPLE.** The Deligne-Mumford moduli spaces of stable curves of genus $g$ with $n + 1$ points. The operadic composite maps are defined by intersecting curves along their marked points.
Configuration spaces vs Moduli spaces

Little disk operad ↔ configuration spaces operad.

Let \( \text{Conf}^n(C) \) denote the configuration space of \( n \) marked points on \( C \). We have that:

\[
\text{Conf}^n(C) \cong \text{Conf}^{n+1}(\mathbb{P})
\]

Taking the quotient by the action of \( \text{PGL}_2(C) \), we have:

\[
\overline{M}_{0,n} \cong \overline{\text{Conf}}^{n+1}(\mathbb{P})/\text{PGL}_2(C),
\]

where \( \overline{M}_{0,n} \) is the compactified moduli space of genus 0 curves with marked points.

Example of operads: the genus zero modular operad
**Figure**: Deligne-Mumford moduli spaces, figure from S. Devadoss, *Tessellations of Moduli spaces and the Mosaic operad*
Quadratic algebras category

Let $k$ be a (commutative) field of char. 0. Consider a category of vector spaces over $k$.

**Definition** A *quadratic algebra* is a graded $k$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$, where $A_0 = k$, $A_1$ is a finite dimensional subspace generating $A$, and such that an appropriate subspace $R(A) \subset A \otimes^2$ generates the ideal of all relations between elements of $A_1$. 
$A$ is given together with the surjective morphism of the tensor algebra of $A_1$ to $A$, whose kernel in the component of degree $d \geq 2$ equals

$$\sum_{i + k = d - 2} A_1 \otimes_i \otimes_k R(A) \otimes_k A_1 \otimes^k.$$ 

- We write $A \leftrightarrow (A_1, R(A))$. 

Example of operads: the genus zero modular operad
• Quadratic algebras are objects of the category $\text{QA}$,
• morphisms $A \to B$ can be described as linear maps $f : A_1 \to B_1$ such that $(f \otimes f)(R(A)) \subset R(B)$.

Whenever we are dealing only with $\text{QA}$ as a category, we may simply denote its objects $(A_1, R(A))$.

♣ There is also the natural functor $\text{QA} \to \text{Lin}_k$ (where $\text{Lin}_k$ is the category of finite dimensional linear spaces over $k$). It is given by $A \to A_1$. 
An operad $P$ is a tensor functor between symmetric monoidal categories $(\Gamma, \sqcup) \rightarrow (QA, \otimes)$ where $\Gamma$ is a category of labelled (finite) graphs with disjoint union; tensor product in $QA$ is defined as:

$$(A_1, R(A)) \otimes (B_1, R(B)) := (A_1 \oplus B_1, R(A) \oplus [A_1, B_1] \oplus R(B)).$$
Operad with target category $(QA, \otimes)$

The data completely determining such an operad is the set of morphisms in the target category $(QA, \otimes)$:

$$P(k) \otimes P(m_1) \otimes P(m_2) \otimes \cdots \otimes P(m_k) \to P(n), \quad n = m_1 + \cdots + m_k,$$

indexed by unshuffles of $\{1, 2, \ldots, n\}$.
Genus 0 modular operad

We consider the shuffle operad in the category $\mathbf{QA}$: the genus 0 modular (co)operad $P$.

The component of arity $n$, for $n \geq 2$ of $P$, is the cohomology ring $P(n) := H^*(\overline{M}_{0,n+1}, \mathbb{Q})$, where $\overline{M}_{0,n+1}$ is the moduli space (projective manifold) parametrising stable curves of genus zero with $n + 1$ labelled points. Component of arity 1 is $\mathbb{Q}$.
Genus 0 modular operad

Structure morphisms (cooperadic comultiplications):

\[ P(m_1 + m_n + \cdots + m_k) \to P(k) \otimes P(m_1) \otimes P(m_2) \otimes \cdots \otimes P(m_k) \]

are maps induced by the maps of moduli spaces defined point-wise by a glueing of the respective stable curves:

\[ \overline{M}_{0,k+1} \times \overline{M}_{0,m_1+1} \times \overline{M}_{0,m_2+1} \cdots \times \overline{M}_{0,m_k+1} \to \overline{M}_{0,m_1+\cdots+m_k+1} \]
Remark

There is another operad $G$ whose components of every arity are quadratic algebras as well. It encodes Gerstenhaber algebras (Loday–Vallette, pp. 506 and 536).

Each $G(n)$ can be represented as the homology ring of the Fulton–MacPherson compactification of the space of configurations of $n$ points in $R^2$. 
Operad characterised by the category of algebras that it classifies

The operad $P$ produces algebras endowed with infinitely many multilinear operations satisfying infinitely many “multicommutativity” properties.

- Let $L$ be a linear space with symmetric even non-degenerate scalar product $h$.
  An action of $P$ upon it induces upon $L$ the hypercommutative (or hyperCom) algebra (see next slide).
4.4.1. Definition. A structure of cyclic hyperCom-algebra on $(L, g)$ is a sequence of polylinear multiplications

$$\circ_n : L^\otimes n \to L, \circ_n(\gamma_1 \otimes \cdots \otimes \gamma_n) =: (\gamma_1, \ldots, \gamma_n), \quad n \geq 2$$

satisfying three axioms:

(i) Commutativity = $S_n$-symmetry;

(ii) Cyclicity: $h((\gamma_1, \ldots, \gamma_n), \gamma_{n+1})$ is $S_{n+1}$-symmetric;

(iii) Associativity: for any $m \geq 0$, $\alpha, \beta, \gamma, \delta_1, \ldots, \delta_m$

$$\sum_{\{1, \ldots, m\} = S_1 \sqcup S_2} \pm((\alpha, \beta, \delta_i \mid i \in S_1), \gamma, \delta_j \mid j \in S_2) =$$

$$\sum_{\{1, \ldots, m\} = S_1 \sqcup S_2} \pm(\alpha, \delta_i \mid i \in S_1), \beta, \gamma, \delta_j \mid j \in S_2))$$

with usual signs from superalgebra.

(iv) (Optional) identity Data and Axiom: $e \in L_{even}$ satisfying

$$(e, \gamma_1, \ldots, \gamma_n) = \gamma_1$$ for $n = 1$; $0$ for $n \geq 2.$