Example of operads: the genus zero modular operad

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MAX-PLANCK-GESELLSCHAFT

Definition with vector spaces

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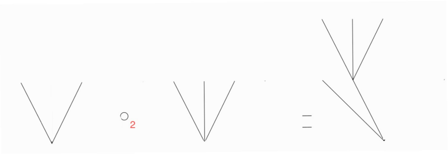

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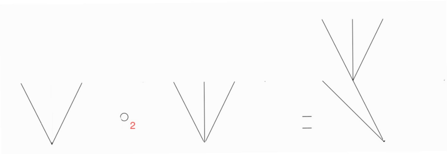

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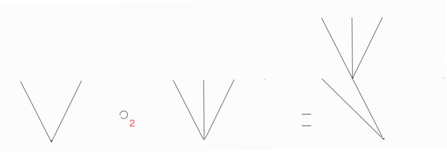

- a unit morphism $\eta : \mathbf{1} \to P(1)$
- $+$ satisfying some axioms (equivariance, unit, associativity).

Operads can be applied everywhere ...
... as long as you have a

symmetric monoidal category.

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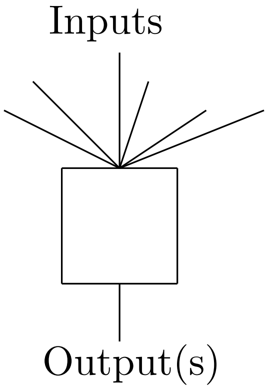
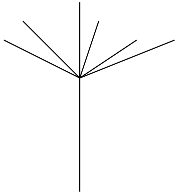
4. natural isomorphisms $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$
5. coherence axioms,
6. and symmetry isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ such that $c_{X,Y} c_{Y,X} = id$.

We can think of an n-ary operation as a little black box with n wires coming in and one wire coming out:



Shrink the black box to a point, you obtain this graph $\natural$:



Trees for operads

Tree T**:**

non-empty, connected graph.
No loops.
Can be oriented.

Property: At each vertex there exists at least one incoming edge; exactly **one** outgoing edge.

External edge: bounded by a vertex (one end only).

Internal edges: those bounded by vertices at both ends (all edges that are nor external)

Any tree has:

- a **unique outgoing** external edge, called the **output (or the root)** of the tree,
- **several ingoing** edges, called **inputs** or leaves of the tree.

Similarly, the edges going in and out of a vertex v of a tree will be referred to as inputs and outputs at v.

Operadic zoo

Algebras:

- **Operad**

Graphs

- **Rooted trees**

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Algebras:

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- connected directed graphs w/o directed loops or parallel edges

Operadic zoo

Algebras:

- **Operad**
- cyclic operads
- k-modular
- dioperads
- properads

Graphs

- **Rooted trees**
- trees
- connected + orientation + on set of edges + genus marking
- connected directed graphs w/o directed loops or parallel edges
- connected directed graphs w/o directed

Borisov–Manin's generalized operad definition

Definition [Borisov–Manin]

Operads of various types are certain functors from a category of labeled graphs Γ *to a symmetric monoidal category* $(G, \otimes)$ *which will be called ground category.*

Example

The simplest example is that of finite-dimensional vector spaces over a field, or that of finite complexes of such spaces.

N.B: The word 'operad' is in the wide sense (i.e.May and Markl operads, cyclic operads, modular operads, PROPS, properads, dioperads etc.).

Operadic zoo:
What kind of operadic creatures can we find?

Modular operad. **No distinction** between *inputs* and *outputs*.

EXAMPLE. The Deligne-Mumford moduli spaces of stable curves of genus g with $n+1$ points. The operadic composite maps are defined by intersecting curves along their marked points.

Configuration spaces vs Moduli spaces

Little disk operad $\leftrightarrow$ configuration spaces operad.

Let $Conf_n(\mathbb{C})$ denote the configuration space of n marked points on $\mathbb{C}$. we have that:

$$Conf_n(\mathbb{C}) \cong Conf_{n+1}(\mathbb{P})$$

Taking the quotient by the action of $PGL_2(\mathbb{C})$, we have:

$$\overline{M}_{0,n} \cong \overline{Conf}_{n+1}(\mathbb{P})/PGL2(\mathbb{C}),$$

where $\overline{M}_{0,n}$ is the compactified moduli space of genus 0 curves with marked points.

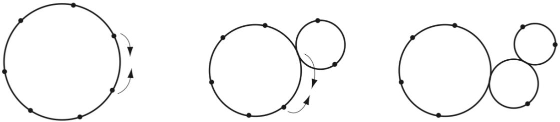

Figure: Deligne-Mumford moduli spaces, figure from S. Devadoss, *Tessellations of Moduli spaces and the Mosaic operad*

Quadratic algebras category

Let k be a (commutative) field of char. 0. Consider a category of vector spaces over k.

Definition A *quadratic algebra* is a graded k-algebra $A = \oplus_{i=0}^{\infty} A_i$, where $A_0 = k$, A_1 is a finite dimensional subspace generating A, and such that an appropriate subspace $R(A) \subset A^{\otimes 2}$ generates the ideal of all relations between elements of A_1.

A is given together with the surjective morphism of the tensor algebra of A_1 to A, whose kernel in the component of degree $d \geq 2$ equals

$$\sum_{i+k=d-2} A_1^{\otimes i} \otimes_k R(A) \otimes_k A_1^{\otimes k}.$$

- We write $A \leftrightarrow (A_1, R(A))$.

QA category

- Quadratic algebras are objects of the category **QA**,
- morphisms $A \to B$ can be described as linear maps

$f : A_1 \to B_1$ such that $(f \otimes f)(R(A)) \subset R(B)$.

Whenever we are dealing only with **QA** as a category, we may simply denote its objects $(A_1, R(A))$.

♣ There is also the natural functor **QA** $\to Lin_k$ (where Lin_k is the category of finite dimensional linear spaces over k). It is given by $A \to A_1$.

Definition by Borisov–Manin

An operad P is a tensor functor between symmetric monoidal categories $(\Gamma, \sqcup) \to (\mathbf{QA}, \otimes)$ where Γ is a category of labelled (finite) graphs with disjoint union; tensor product in $\mathbf{QA}$ is defined as:

$$(A_1, R(A)) \otimes (B_1, R(B)) := (A_1 \oplus B_1, R(A) \oplus [A_1, B_1] \oplus R(B)).$$

Operad with target category $(\mathbf{QA}, \otimes)$

The data completely determining such an operad is the set of morphisms in the target category $(\mathbf{QA}, \otimes)$:

$$P(k)\otimes P(m_1)\otimes P(m_2)\otimes\cdots\otimes P(m_k) \to P(n), \quad n = m_1+\cdots+m_k,$$

indexed by unshuffles of $\{1, 2, ..., n\}$

Genus 0 modular operad

We consider the shuffle operad in the category **QA** : the genus 0 modular (co)operad P.

The component of arity n, for $n \geq 2$ of P, is the cohomology ring $P(n) := H^*(\overline{M}_{0,n+1}, \mathbf{Q})$, where $\overline{M}_{0,n+1}$ is the moduli space (projective manifold) parametrising stable curves of genus zero with $n+1$ labelled points. Component of arity 1 is **Q.**

Genus 0 modular operad

Structure morphisms (cooperadic comultiplications):

$$P(m_1 + m_n + \cdots + m_k) \to P(k) \otimes P(m_1) \otimes P(m_2) \otimes \cdots \otimes P(m_k)$$

are maps induced by the maps of moduli spaces defined point-wise by a glueing of the respective stable curves:

$$\overline{M}_{0,k+1} \times \overline{M}_{0,m_1+1} \times \overline{M}_{0,m_2+1} \cdots \times \overline{M}_{0,m_k+1} \to \overline{M}_{0,m_1+\cdots+m_k+1}$$

Remark

There is another operad G whose components of every arity are quadratic algebras as well. It encodes Gerstenhaber algebras (Loday–Vallette, pp. 506 and 536).

Each $G(n)$ can be represented as the homology ring of the Fulton–MacPherson compactification of the space of configurations of n points in R^2.

Operad characterised by the category of algebras that it classifies

The operad P produces algebras endowed with **infinitely many multilinear operations** satisfying infinitely many **"multicommutativity" properties**.

• Let L be a linear space with symmetric even non–degenerate scalar product h.
An action of P upon it induces upon L the hypercommutative (or hyperCom) algebra (see next slide).

4.4.1. Definition. *A structure of cyclic hyperCom–algebra on* (L, g) *is a sequence of polylinear multiplications*

$$\circ_n : \ L^{\otimes n} \to L, \ \circ_n(\gamma_1 \otimes \cdots \otimes \gamma_n) =: (\gamma_1, \ldots, \gamma_n), \ n \geq 2$$

satisfying three axioms:

(i) Commutativity $= \mathbf{S}_n$*–symmetry;*

(ii) Cyclicity: $h((\gamma_1, \ldots, \gamma_n), \gamma_{n+1})$ *is* $\mathbf{S}_{n+1}$*–symmetric;*

(iii) Associativity: for any $m \geq 0$, $\alpha, \beta, \gamma, \delta_1, \ldots, \delta_m$

$$\sum_{\{1,\ldots,m\}=S_1 \amalg S_2} \pm((\alpha, \beta, \delta_i \,|\, i \in S_1), \gamma, \delta_j \,|\, j \in S_2) =$$

$$\sum_{\{1,\ldots,m\}=S_1 \amalg S_2} \pm(\alpha, \delta_i \,|\, i \in S_1), \beta, \gamma, \delta_j \,|\, j \in S_2))$$

with usual signs from superalgebra.

(iv) (Optional) identity Data and Axiom: $e \in L_{even}$ *satisfying*

$$(e, \gamma_1, \ldots, \gamma_n) = \gamma_1 \text{ for } n = 1; \ 0 \text{ for } n \geq 2.$$